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Radiation damping of inertial oscillations in the upper ocean

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Turbulent motions within the wind-mixed layer, which is advected by near-surface inertial oscillations, excite internal gravity waves in the underlying ocean layers. Momentum transport in the radiated wave field results in a drag force on the inertial currents. Because the magnitude of the inertial currents is large compared with the turbulence intensity, the resultant rate of dissipation of inertial oscillation energy is approximately equal to the energy flux in the radiated wave field. Using linear internal wave theory, asymptotic results are derived for the energy flux in terms of the Brunt–Väisälä frequency N below the mixed layer, the magnitude U_0 of the inertial current, the integral length scale l of the mixed-layer turbulence and the mean-square displacement $\langle \zeta_0^2 \rangle$ of the base of the mixed layer. For representative conditions, we estimate an energy flux of 1–10 erg/cm^2 s into relatively short (wavelength of order $2\pi U_0/N$) high frequency (of order, but less than, N) internal waves. The resultant decay times for inertial oscillation energy range from a day to a week or so, in agreement with reported observations on the decay of inertial oscillations in the upper ocean. The estimated energy flux is comparable in magnitude to estimates for other internal wave generation mechanisms, indicating that, in addition to being a significant sink of inertial energy, this process may locally represent a significant source of internal wave energy in the open ocean.

1. Introduction

Inertial oscillations, i.e. rotary currents with frequency close to the local inertial frequency f (at latitude Λ, the inertial period $2\pi/f$ is 12 h/sin Λ), are a characteristic feature of current measurements in the ocean. The most striking aspect of such currents is their transience. The oscillations are highly intermittent, in many cases persisting for only a few cycles (see Webster 1968, for example). It is probable that in many instances inertial oscillations are initiated by fluctuations in the wind stress via geostrophic adjustment processes.† This generation mechanism has been studied by several authors, including Pollard (1970), Gonella (1971) and Krauss (1972), and comparisons between theory and observation are generally quite good (Pollard & Millard 1970; Kundu 1976). Although the generation of inertial oscillations would thus appear to be fairly well understood, the same is not true of their dissipation. To date, no satisfactory explanation of the observed transience of inertial oscillations has been proposed (see Smith 1973).

† Blumen (1972) has made a comprehensive review of geostrophic adjustment processes.

Wind-stress forcing as a generation process is restricted for the most part to the wind-mixed layer below the ocean surface. Although in due course some of the near-inertial energy leaks into the underlying ocean layers as a result of the propagation characteristics of inertial–gravity waves (Kroll 1975), this is a relatively slow process. The bulk of the available observational evidence (see, for example, Webster 1968; Pollard 1970; Pollard & Millard 1970; Gonella 1971; Pollard, Rhines & Thompson 1973; Halpern 1974; Hayes & Halpern 1976) indicates that for the most part wind-generated inertial oscillations are slab-like, advecting the wind-mixed layer as a whole, with little direct influence on the motion of the underlying ocean layers. However, because the mixed layer is turbulent, an indirect coupling may exist between near-surface inertial oscillations and higher frequency internal gravity wave motions in the underlying ocean layers. Lumps and bumps in the base of the mixed layer associated with the turbulent eddies are advected by the inertial oscillations and residual turbulent motions and disturb the underlying stratified ocean layers, resulting in the generation of internal gravity waves. The wave field radiates momentum and energy from the mixed layer. The energy flux is related to the rate of working against normal and tangential forces associated with the interaction of turbulent eddies with the stratified region below. The normal forces relate to the time-dependent nature of the turbulence, while the tangential forces relate to the advection of the lumps and bumps by the inertial oscillations and by other turbulent eddies. If the magnitude of the inertial current is large compared with the turbulence intensity, as is usually the case, then most of the radiated energy will be derived from the inertial oscillations. In the analysis which follows, we investigate internal wave radiation from the mixed layer and its relationship to near-surface inertial oscillations. The results reported here indicate that, for representative oceanic conditions, internal wave radiation may account for the observed transience of near-surface inertial oscillations.

The theory of internal wave radiation from a turbulent layer has been considered by several authors, including Townsend (1965, 1966, 1968), Tolstoy (1973) and Tolstoy & Miller (1975). Although application of the existing theory has been restricted to meteorological contexts, the authors have noted that analogous phenomena may occur in the ocean, and that elements of the theory may be applicable in the oceanic case. However, owing to peculiarities in the oceanic system, the theory developed in these earlier studies must be modified and extended for oceanic application. Whereas the earlier theories apply to a situation in which relative motions between the turbulent and non-turbulent layers are steady, relative motions between the surface mixed layer and the underlying upper ocean layers are time dependent, being associated with the wind-driven inertial oscillations. Thus we are led to consider the problem of internal wave radiation from a turbulent layer which is advected by a current whose components vary harmonically in time. The resultant modifications to the existing theory are similar to those which arise in the extension of classical lee-wave theory to oscillatory flows (see Bell 1975*a*, *b*). Owing to the disparity between the inertial frequency and other natural frequencies of the system, it turns out that a quasi-steady approximation is valid. The general solution for time-dependent currents is shown to reduce in the limit $f \rightarrow 0$ to that which would be obtained by considering wave generation with a steady advection current and then averaging over all possible current directions. Hence we could start with Townsend's (1968, equation 2.14) result, average over all possible current directions, then apply the result to the problem at hand. However, since the general theory for harmonic currents is interesting in its own right, we develop the general theory then pass to the quasi-steady limit.

The fundamental theory is developed in §2. We adopt Townsend's approach, representing the effect of the mixed-layer turbulence by vertical displacements of the base of the mixed layer. The theory is developed for a uniformly stratified, bottomless ocean, thereby isolating the essential physics of the problem. With regard to oceanographic applications, it is readily shown that if the first-mode, long-wave speed appropriate to the real ocean is large compared with the magnitude of the inertial currents then this semi-infinite approximation is in fact valid, at least in so far as the energetics are concerned. In §3, we explore the characteristics of advected-mixed-layer turbulence and introduce some simplifications into the theoretical results. The question of Kelvin–Helmholtz instability at the base of the mixed layer is not addressed directly. This is not to say that structures resulting from such instabilities cannot or will not exist. However, available information indicates that the dominant (at least observationally) structures are elongated vortices which are aligned more or less parallel to the prevailing wind direction, rather than exhibiting any preferred orientation with respect to the direction of the shear at the base of the mixed layer, which will vary depending on the phase of the inertial oscillation. The theory is developed with this observation in mind. In §4, we consider the radiative damping of near-surface inertial oscillations. Significant simplifications are obtained by taking advantage of certain properties of the turbulence and disparities in length and time scales, and we are able to obtain simple asymptotic expressions for the major theoretical results in terms of integral properties of the turbulence. Some properties of the radiated internal wave field are considered in §5. The significance of the analytical results is discussed briefly in §6. Precise evaluation of the theory is somewhat hindered by our imprecise knowledge of some of the properties of mixed-layer turbulence. However, it would appear that, for representative oceanic conditions, we may expect an energy flux of 1–10 $\mathrm{erg/cm^2 s}$ into relatively short (wavelength of order several hundred metres or less), high frequency (of the order of, but less than, the Brunt–Väisälä frequency) internal waves. Since the bulk of this energy is derived from the inertial oscillations, we estimate decay times for the magnitude of the inertial currents ranging from a day to a week or so, in agreement with reported observations. Furthermore, our estimate of the energy flux is comparable in magnitude to estimates for other generation mechanisms (reviewed by Müller & Olbers 1975), indicating that, in addition to being a significant sink of inertial energy, this process may locally represent a significant source of internal wave energy in the open ocean.

2. Internal wave generation

Consider a region of uniformly stratified fluid underlying a turbulent mixed layer. With the effects of rotation included, small amplitude internal gravity waves in the lower, stratified layer are governed by the equation

$$(D^2+N^2)\nabla^2\eta+(D^2+f^2)\eta_{zz}=0, \tag{2.1}$$

where η is the vertical displacement amplitude of the wave motion, $D=\partial/\partial t$, $\nabla^2=\partial^2/\partial x^2+\partial^2/\partial y^2$, f is the local inertial frequency (Coriolis parameter) and N is the

Brunt–Väisälä frequency, defined in terms of the vertical density gradient by

$$N^2 = -\frac{g}{\rho}\frac{d\rho}{dz}, \tag{2.2}$$

where g is the acceleration due to gravity. In modelling the effect of the turbulent mixed layer on the stratified region, we follow the approach developed by Phillips (1955) for determining the irrotational motion outside a free turbulent boundary and later extended by Townsend (1965, 1966, 1968) to include the effects of stratification. The effect of the turbulence is modelled by a random displacement of the base of the mixed layer, which serves to force the internal wave motion in the stratified region below.

Considering first an isolated disturbance, the boundary conditions to be imposed on the governing equation (2.1) are that η be Fourier transformable in x, y and t and that

$$\eta(x, y, 0, t) = \zeta(x, y, t), \tag{2.3}$$

where $z = 0$ at the base of the mixed layer and $\zeta(x, y, t)$ is the turbulent displacement of the base of the mixed layer. The solution is rendered determinate by a condition of downward radiation of wave energy. This is the lower boundary condition appropriate to a uniformly stratified, bottomless ocean. No internal wave energy is reflected back up towards the mixed layer. The opposite extreme is the waveguide model ocean, in which internal wave energy is trapped by a region of strong stratification or by the effect of a rigid boundary. The real ocean is a waveguide, although dissipative processes may make it appear radiative in so far as certain generation mechanisms are concerned. However, it can be shown that, if the limiting phase speed of trapped internal wave modes is large compared with the magnitude of the inertial currents advecting the mixed layer, then the radiative solution will yield a good approximation for the flux of energy into the internal wave field. This point is illustrated in figure 1, where the ratio of the energy flux in a uniformly stratified waveguide model ($\mathscr{F}_w$) to that in a radiative model ($\mathscr{F}_R$) is shown as a function of the ratio of the limiting phase speed for guided waves (C_0) to the magnitude of the inertial oscillation (U_0). For $C_0/U_0 < 1$, the waveguide flow is supercritical, and the efficiency of internal wave generation is decreased. The peaks at integral values of C_0/U_0 correspond to the limiting phase speeds of the respective waveguide modes and represent tuned responses to the forcing with speed $U_0 = C_0/n$. As the flow becomes more and more subcritical (C_0/U_0 increasing), the efficiency of internal wave generation of the waveguide approaches that of the semi-infinite ocean. Garrett & Munk (1972) give dispersion relations for a model ocean. The limiting phase speed in their model is 2·3 m/s. Assuming that this is representative of the real ocean, it is clear that, for inertial currents of less than a metre per second or so, the radiative approximation should be adequate.

Introducing the Fourier transform

$$\hat{\eta}(\kappa, \lambda, \omega; z) = \iiint_{-\infty}^{\infty} \eta(x, y, z, t) \exp\left[-i(\kappa x + \lambda y - \omega t)\right] dx\, dy\, dt \tag{2.4}$$

reduces the governing equation (2.1) to

$$\hat{\eta}'' + k^2 \frac{N^2 - \omega^2}{\omega^2 - f^2}\hat{\eta} = 0, \tag{2.5}$$

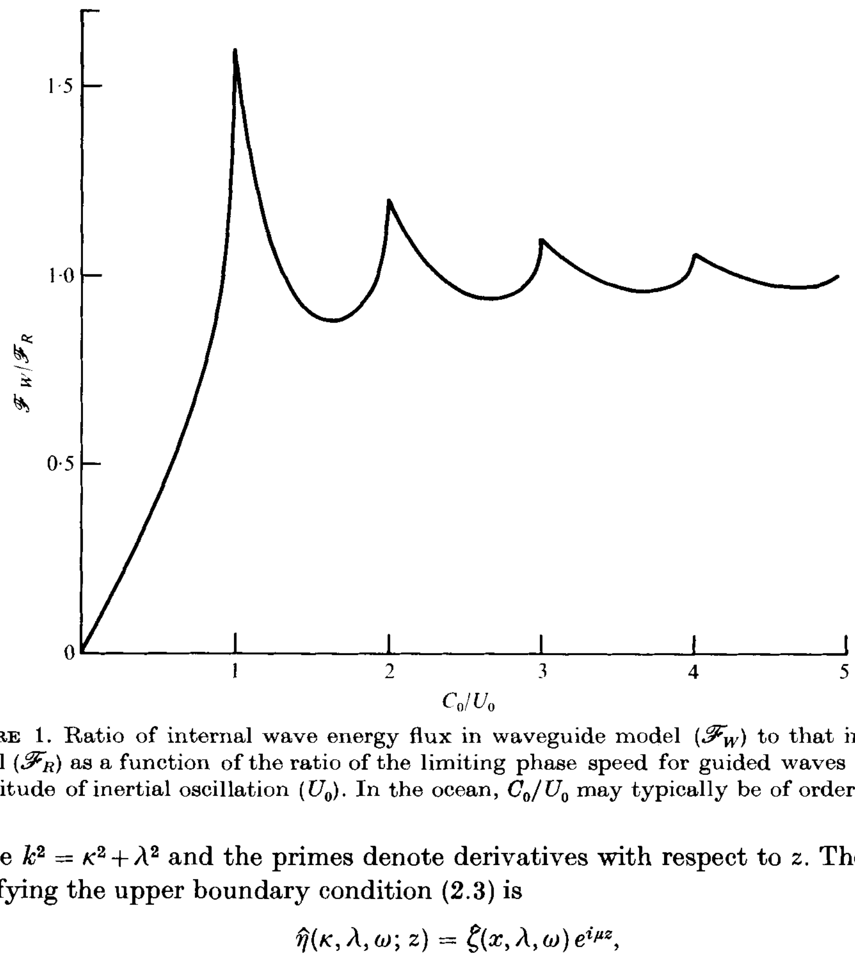

FIGURE 1. Ratio of internal wave energy flux in waveguide model ($\mathscr{F}_W$) to that in radiative model ($\mathscr{F}_R$) as a function of the ratio of the limiting phase speed for guided waves (C_0) to the magnitude of inertial oscillation (U_0). In the ocean, C_0/U_0 may typically be of order 10.

where $k^2 = \kappa^2 + \lambda^2$ and the primes denote derivatives with respect to z. The solution satisfying the upper boundary condition (2.3) is

$$\hat{\eta}(\kappa, \lambda, \omega; z) = \hat{\zeta}(x, \lambda, \omega)\, e^{i\mu z}, \tag{2.6}$$

where

$$\mu^2 = k^2(N^2 - \omega^2)/(\omega^2 - f^2). \tag{2.7}$$

There are two parameter regimes for the solution. If $f^2 < \omega^2 < N^2$, μ is real and the solution (2.6) is oscillatory in z, corresponding to internal wave motions. Otherwise, μ is imaginary and the solution is monotonic in z. In this evanescent regime, the choice

$$\mu = -ik\left|\frac{N^2 - \omega^2}{\omega^2 - f^2}\right|^{\frac{1}{2}} \tag{2.8}$$

ensures the appropriate decay away from the generation region. In the oscillatory regime, the radiation condition requires that the vertical component

$$\gamma = \partial\omega/\partial\mu \tag{2.9}$$

of the group velocity be negative, corresponding to downward propagation of wave energy. From (2.7),

$$\frac{\partial\omega}{\partial\mu} = -\frac{\mu}{\omega}\frac{k^2(N^2 - f^2)}{(k^2 + \mu^2)^2}, \tag{2.10}$$

which is negative if $\operatorname{sgn}\mu = \operatorname{sgn}\omega$. Thus, in the internal wave regime, the choice

$$\mu = \operatorname{sgn}(\omega)\, k\left(\frac{N^2 - \omega^2}{\omega^2 - f^2}\right)^{\frac{1}{2}} \tag{2.11}$$

renders the solution determinate. The complete solution is given by the inverse transform

$$\eta(x,y,z,t) = \frac{1}{(2\pi)^3}\iint\int_{-\infty}^{\infty} \hat{\zeta}(\kappa,\lambda,\omega)\exp[i(\kappa x+\lambda y+\mu z-\omega t)]\,d\kappa\,d\lambda\,d\omega, \tag{2.12}$$

where

$$\hat{\zeta}(\kappa,\lambda,\omega) = \iint\int_{-\infty}^{\infty} \zeta(x,y,t)\exp[-i(\kappa x+\lambda y-\omega t)]\,dx\,dy\,dt \tag{2.13}$$

is the Fourier transform of the vertical displacement associated with the disturbance at the base of the mixed layer.

Mixed-layer turbulence, of course, is not an isolated disturbance. It is more properly represented as a stationary random process, hence so is the radiated wave field. Since we are dealing with linear theory, the appropriate solution entails second-moment (spectral) representations of the turbulence and the wave field. This could be accomplished by superposing the effects of an appropriate random distribution of isolated disturbances. Alternatively, we may simply note that the solution (2.12) represents a linear transformation of the disturbance, so that its spectral function is related to that of the disturbance by

$$\hat{R}_\eta(\kappa,\lambda,\omega;z) = \hat{R}_\zeta(\kappa,\lambda,\omega)\exp[i(\mu-\mu^*)z], \tag{2.14}$$

where μ^* is the complex conjugate of μ, and the spectral functions $\hat{R}_\eta$ and $\hat{R}_\zeta$ are Fourier transforms of correlation functions. In particular,

$$\hat{R}_\zeta(\kappa,\lambda,\omega) = \iint\int_{-\infty}^{\infty} R_\zeta(x,y,t)\exp[-i(\kappa x+\lambda y-\omega t)]\,dx\,dy\,dt, \tag{2.15}$$

where

$$R_\zeta(x,y,t) = \langle\zeta(0,0,0)\,\zeta(x,y,t)\rangle \tag{2.16}$$

is the correlation function which characterizes the turbulence-induced displacements of the base of the mixed layer. The energy density in the radiated wave field may be expressed in terms of the spectral function $\hat{R}_\zeta$. In the internal wave regime, μ is real, so that $\hat{R}_\eta = \hat{R}_\zeta$. The mean-square vertical displacement in the wave field is then given by

$$\langle\eta^2\rangle = \frac{1}{(2\pi)^3}\int \hat{R}_\zeta(\kappa,\lambda,\omega)\,d\kappa\,d\lambda\,d\omega, \tag{2.17}$$

where the integral extends over all wavenumbers and all frequencies such that $f^2 < \omega^2 < N^2$. The potential energy density is then $\frac{1}{2}\rho_0 N^2\langle\eta^2\rangle$. Invoking the appropriate partition of energy for internal inertial–gravity waves (see Fofonoff 1969), the total energy density in the radiated wave field is

$$\mathscr{E} = \frac{\rho_0}{(2\pi)^3}\int \omega^2\hat{R}_\zeta(\kappa,\lambda,\omega)\,\frac{N^2-f^2}{\omega^2-f^2}\,d\kappa\,d\lambda\,d\omega. \tag{2.18}$$

The flux of energy into the internal wave field is obtained by multiplying the spectral function $\hat{R}_\zeta$ in (2.18) by the magnitude of the group velocity given by (2.9) and (2.10):

$$\mathscr{F} = \frac{\rho_0}{(2\pi)^3}\int \frac{|\omega|}{k}[(N^2-\omega^2)(\omega^2-f^2)]^{\frac{1}{2}}\,\hat{R}_\zeta(\kappa,\lambda,\omega)\,d\kappa\,d\lambda\,d\omega. \tag{2.19}$$

The results (2.18) and (2.19) characterize the internal wave generation process. We may recover Townsend's (1968, equation 2.14) result from the energy flux due to internal wave generation by a turbulent boundary layer in a stratified atmosphere by setting $f = 0$ and approximating the spectral function $\hat{R}_\zeta$ by the form

$$\hat{R}_\zeta = 2\pi \hat{R}_0(\kappa, \lambda)\, \delta(\omega - \mathbf{k}\,.\,\mathbf{U}), \tag{2.20}$$

where $\mathbf{U}$ is a constant advection velocity, so that (2.20) represents a steady disturbance field which is advected relative to the stratified layer at a constant speed.

3. Advected-mixed-layer turbulence

If the mixed layer is moving relative to the underlying upper ocean layers, the effect of advection must be incorporated in the disturbance correlation function $R_\zeta(x, y, t)$. Townsend and others have considered the case of steady advection. Here we extend the theory to a class of time-dependent motions. For general time-dependent motion, the disturbance correlation function may be expressed in the form

$$R_\zeta(x, y, t) = R_{\zeta 0}\left(x - \int U\, dt, y - \int V\, dt, t\right), \tag{3.1}$$

where $U(t)$ and $V(t)$ are the x and y components of a general time-dependent near-surface current and $R_{\zeta 0}(x, y, t)$ is the correlation function seen by an observer moving with the mixed layer, i.e. the 'pure turbulence' correlation function. The spectral function is then given by

$$\hat{R}_\zeta(\kappa, \lambda, \omega) = \iiint_{-\infty}^{\infty} R_{\zeta 0}(x, y, t) \exp\{-i[\kappa x + \lambda y - \sigma(t)]\}\, dx\, dy\, dt, \tag{3.2}$$

where
$$\sigma(t) = \omega t - \kappa \int U\, dt - \lambda \int V\, dt. \tag{3.3}$$

For pure inertial oscillations,

$$U(t) = U_0 \cos(ft + \alpha), \quad V(t) = U_0 \sin(ft + \alpha), \tag{3.4}$$

where α is an arbitrary phase. Substituting into (3.3), we have

$$\sigma(t) = \omega t - \frac{kU_0}{f} \sin(\phi - \alpha) - \frac{kU_0}{f} \sin(ft + \alpha - \phi), \tag{3.5}$$

where we have set
$$\kappa = k \cos\phi, \quad \lambda = k \sin\phi. \tag{3.6}$$

Invoking the Neumann expansion

$$\exp\left(-i \frac{kU_0}{f} \sin(ft + \alpha - \phi)\right) = \sum_{n=-\infty}^{\infty} \exp[-in(ft + \alpha - \phi)]\, J_n\left(\frac{kU_0}{f}\right), \tag{3.7}$$

where the J_n are Bessel functions (Watson 1966, §2.22), we may substitute into (3.2) and obtain

$$\hat{R}_\zeta(\kappa, \lambda, \omega) = \exp\left(-i \frac{kU_0}{f} \sin(\phi - \alpha)\right) \sum_{n=-\infty}^{\infty} J_n\left(\frac{kU_0}{f}\right) \exp[in(\phi - \alpha)]\, \hat{R}_{\zeta 0}(\kappa, \lambda, \omega - nf), \tag{3.8}$$

where now

$$\hat{R}_{\zeta 0}(\kappa, \lambda, \omega) = \iiint_{-\infty}^{\infty} R_{\zeta 0}(x, y, t) \exp[-i(\kappa x + \lambda y - \omega t)]\, dx\, dy\, dt \tag{3.9}$$

is the spectral function for the turbulence displacement field. Averaging over phase, we finally obtain the expression

$$\hat{R}_\zeta(\kappa,\lambda,\omega) = \sum_{n=-\infty}^{\infty} J_n^2\left(\frac{kU_0}{f}\right)\hat{R}_{\zeta 0}(\kappa,\lambda,\omega-nf) \tag{3.10}$$

for the spectral function which drives the wave motion (see Watson 1966, §2.2).

Substitution of (3.10) into (2.19) yields the expression

$$\mathscr{F} = \frac{\rho_0}{(2\pi)^3}\int\frac{|\omega|}{k}[(N^2-\omega^2)(\omega^2-f^2)]^{\frac{1}{2}}\Sigma J_n^2\left(\frac{kU_0}{f}\right)\hat{R}_{\zeta 0}(\kappa,\lambda,\omega-nf)\,d\kappa\,d\lambda\,d\omega \tag{3.11}$$

for the energy flux. As in the earlier studies of wave generation by oscillatory flows (Bell 1975*a, b*), the generation process samples all the sum and difference frequencies $(\omega \pm nf)$. The sum in (3.11) is readily transformed to an integral by noting that

$$\sum_{n=-\infty}^{\infty} J_n^2\left(\frac{kU_0}{f}\right)e^{inft} = J_0\left(\frac{2kU_0}{f}\sin\frac{ft}{2}\right) \tag{3.12}$$

(Oberhettinger 1973, equation 4.29). Then, with

$$\hat{R}_{\zeta 0}(\kappa,\lambda,\omega) = \int_{-\infty}^{\infty}\tilde{R}_{\zeta 0}(\kappa,\lambda,t)\,e^{-i\omega t}\,dt, \tag{3.13}$$

we have

$$\sum_{n=-\infty}^{\infty} J_n^2\left(\frac{kU_0}{f}\right)\hat{R}_{\zeta 0}(\kappa,\lambda,\omega-nf) = \int_{-\infty}^{\infty}\tilde{R}_{\zeta 0}(\kappa,\lambda,t)\,J_0\left(\frac{2kU_0}{f}\sin\frac{ft}{2}\right)e^{-i\omega t}\,dt. \tag{3.14}$$

Appreciable contributions to the integral are restricted to $t \lesssim \theta$, where θ is the integral time scale of the turbulence. With $\theta f \ll 1$, we then have

$$\sum_{n=-\infty}^{\infty} J_n^2\left(\frac{kU_0}{f}\right)\hat{R}_{\zeta 0}(\kappa,\lambda,\omega-nf) \sim \int_{-\infty}^{\infty}\tilde{R}_{\zeta 0}(\kappa,\lambda,t)\,J_0(kU_0t)\,e^{-i\omega t}\,dt. \tag{3.15}$$

By the convolution theorem, the integral in (3.15) corresponds to a smoothing of the spectrum $\hat{R}_{\zeta 0}(\kappa,\lambda,\omega)$. Invoking the integral representation for J_0,

$$J_0(kU_0t) = \frac{1}{\pi}\int_0^{\pi}\exp(ikU_0t\cos\phi)\,d\phi \tag{3.16}$$

(Watson 1966, §2.2), we have

$$\int_{-\infty}^{\infty}\tilde{R}_{\zeta 0}(\kappa,\lambda,t)\,J_0(kU_0t)\,e^{-i\omega t}\,dt = \frac{1}{\pi}\int_0^{\pi}\hat{R}_{\zeta 0}(\kappa,\lambda,\omega-kU_0\cos\phi)\,d\phi. \tag{3.17}$$

The sum in (3.11) has thus been transformed to a simple spectral smoothing operation:

$$\mathscr{F} = \frac{\rho_0}{8\pi^4}\int k^{-1}\omega^2(N^2-\omega^2)^{\frac{1}{2}}\hat{R}_{\zeta 0}(\kappa,\lambda,\omega-kU_0\cos\phi)\,d\phi\,d\omega\,d\kappa\,d\lambda, \tag{3.18}$$

where we have replaced ω^2-f^2 with ω^2, consistent with the limit $\theta f \ll 1$.

Observational evidence relating to mixed-layer dynamics (McLeish 1968; Faller 1971, for example) indicates that the more energetic eddies of mixed-layer turbulence (the so-called Langmuir circulations) tend to be rather elongated, with large length-to-width aspect ratios, and that these features are rather persistent, having lifetimes

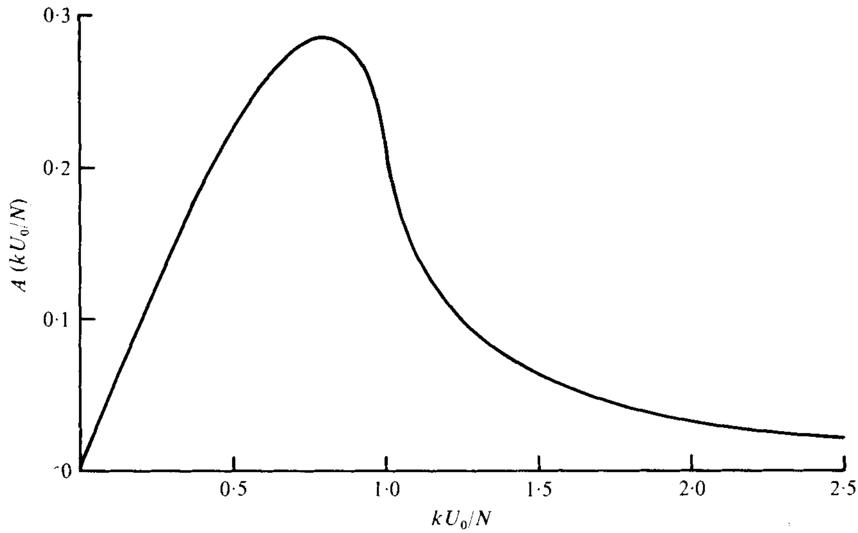

FIGURE 2. The response function $\hat{A}$ as a function of $\beta = kU_0/N$ [see (3.21), (3.22)]. The wave-making response is maximal for wavenumbers slightly less than N/U_0, corresponding to frequencies somewhat less than N in a stationary reference frame.

significantly greater than the characteristic time scale l/u', where l is a characteristic length scale and u' a characteristic velocity scale for the eddies. Representative values of l and u' may be taken as several metres and several centimetres per second, respectively (see §4 below), so that u'/l is characteristically of order $10^{-2}\,\mathrm{s}^{-1}$. This is of comparable order to typical upper-ocean Brunt–Väisälä frequencies. Thus, if θ is the characteristic eddy persistence time, then we may evaluate the energy flux in the asymptotic limit $N\theta \gg 1$. In this case, the frequency bandwidth of $\hat{R}_{\zeta 0}$ is small compared with N, and we have that

$$\frac{1}{2\pi}\int \omega^2(N^2-\omega^2)^{\frac{1}{2}}\,\hat{R}_{\zeta 0}(\kappa,\lambda,\omega-kU_0\cos\phi)\,d\omega \sim k^2U_0^2\cos^2\phi(N^2-k^2U_0^2\cos^2\phi)^{\frac{1}{2}}\,\hat{R}_0(\kappa,\lambda), \tag{3.19}$$

where
$$\hat{R}_0(\kappa,\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{R}_{\zeta 0}(\kappa,\lambda,\omega)\,d\omega \tag{3.20}$$

is the Fourier transform of the spatial correlation of the turbulent displacement field. The energy flux may then be represented in the form

$$\mathscr{F} = \frac{\rho_0 N^2 U_0}{(2\pi)^2}\int\!\!\int_{-\infty}^{\infty}\hat{A}\left(\frac{kU_0}{N}\right)\hat{R}_0(\kappa,\lambda)\,d\kappa\,d\lambda, \tag{3.21}$$

where the response function $\hat{A}(kU_0/N)$ is given by

$$\hat{A}(\beta) = \frac{2}{\pi}\beta\int_0^{\phi_0}\sin^2\phi(1-\beta^2\sin^2\phi)^{\frac{1}{2}}\,d\phi \tag{3.22}$$

with $\phi_0 = \frac{1}{2}\pi$ for $kU_0 < N$ and $\sin\phi_0 = N/kU_0$ otherwise. It may readily be verified that this result is identical with that which would be obtained by considering wave generation with a steady advection current and then averaging over all possible current directions. The response function is plotted *vs.* $\beta = kU_0/N$ in figure 2. The response is maximal for wavenumbers slightly less than N/U_0, corresponding to wave frequencies in a stationary reference frame somewhat less than the Brunt–Väisälä frequency.

Further simplification may be introduced by invoking the elongated nature of the eddies. In this case, we may express the spectral function $\hat{R}_0(\kappa, \lambda)$ in the form

$$\hat{R}_0(\kappa, \lambda) \approx 2\pi \hat{R}_0(\kappa)\, \delta(\lambda) \tag{3.23}$$

for purposes of estimating the energy flux. Since the rotary currents sample all directions within an inertial period, we incur no loss of generality by assuming that the long axes of the turbulent eddies are aligned in the y direction. With the spectral function given by (3.23), our expression for the energy flux becomes

$$\mathscr{F} = \frac{\rho_0 N^2 U_0}{\pi} \int_0^\infty \hat{A}\left(\frac{\kappa U_0}{N}\right) \hat{R}_0(\kappa)\, d\kappa \tag{3.24}$$

with $\hat{A}(\kappa U_0/N)$ given by (3.22).

4. Radiation damping

The energy in the internal wave field is a direct result of work done by motions within the mixed layer against buoyancy forces which arise owing to the density stratification in the underlying upper ocean layers. The rate of working per unit surface area is given by

$$\mathscr{W} = \frac{1}{S} \int_S \mathbf{u} \,.\, \mathbf{n} p \, da, \tag{4.1}$$

where $\mathbf{u}$ is the velocity field at the base of the mixed layer, $\mathbf{n}$ is a unit normal to the surface which defines the base of the mixed layer, and p is the pressure perturbation induced by the internal wave field at the base of the mixed layer. The integral extends over an area S. If $\zeta(x, y, t)$ is the vertical displacement of the base of the mixed layer from its equilibrium level $z = 0$, then the normal component of velocity $\mathbf{u} \,.\, \mathbf{n}$ is given by

$$\mathbf{u} \,.\, \mathbf{n} = -\zeta_t / (\zeta_x^2 + \zeta_y^2 + 1)^{\frac{1}{2}}. \tag{4.2}$$

Consistent with the linearization in § 3 above, the denominator in (4.2) may be set equal to 1, and (4.1) may be evaluated at $z = 0$, the mean position of the base of the mixed layer. For stationary random functions, the integral in (4.1) acts as an ensemble average, so that

$$\mathscr{W} = -\langle \zeta_t p \rangle, \tag{4.3}$$

where the angular brackets denote an ensemble average. For small amplitude motions, the perturbation pressure field is related to the displacement amplitude of the internal wave field by

$$\nabla^2 p = \rho_0 (\partial^2/\partial t^2 + f^2)\, \eta_z. \tag{4.4}$$

Invoking the fundamental solution (2.12), we then obtain the formal result

$$\hat{p}(\kappa, \lambda, \omega) = i \operatorname{sgn}(\omega)\, k^{-1} [(N^2 - \omega^2)(\omega^2 - f^2)]^{\frac{1}{2}} \hat{\zeta}(\kappa, \lambda, \omega), \tag{4.5}$$

where a caret denotes Fourier transformation, as before, and we need consider only the internal wave regime $f^2 < \omega^2 < N^2$. The right-hand side of (4.3) may then be expressed in the form

$$\langle \zeta_t p \rangle = -\frac{\rho_0}{(2\pi)^3} \int \frac{|\omega|}{k} [(N^2 - \omega^2)(\omega^2 - f^2)]^{\frac{1}{2}} \hat{R}_\zeta(\kappa, \lambda, \omega)\, d\kappa\, d\lambda\, d\omega, \tag{4.6}$$

where the integral extends over all wavenumbers and frequencies such that

$$f^2 < \omega^2 < N^2,$$

and $\hat{R}_f$ is the spectral function for the displacement field. Comparison of (2.19) and (4.6) indicates that the rate at which energy is radiated away by internal waves is equal to the rate of working by mixed-layer motions, as expected.

Work is done by both the mean (U) and the turbulent ($\mathbf{u}'$) motions and represents a loss of energy by such motions. The energy lost by mixed-layer motions is, of course, gained by the internal wave field. We may partition the energy loss between mean and turbulent motions by invoking the conservation of momentum in the mixed layer. Neglecting internal dissipation and external forcing, this principle leads directly to the equations of mean motion

$$\left.\begin{aligned} \partial U/\partial t - fV &= -(\rho_0 D)^{-1}\langle\zeta_x p\rangle, \\ \partial V/\partial t + fU &= -(\rho_0 D)^{-1}\langle\zeta_y p\rangle, \end{aligned}\right\} \tag{4.7}$$

where D is the mixed-layer depth (note that the mean motion is assumed to be horizontally uniform). Forming an energy equation from (4.7), we then have

$$\partial E/\partial t = -U\langle p\zeta_x\rangle - V\langle p\zeta_y\rangle, \tag{4.8}$$

where

$$E = \tfrac{1}{2}\rho_0 D(U^2 + V^2) \tag{4.9}$$

is the inertial oscillation energy per unit surface area. In a similar manner, we may obtain an energy balance for the mixed-layer turbulence:

$$\partial E'/\partial t = \langle w'p\rangle - \langle u'\zeta_x p\rangle - \langle v'\zeta_y p\rangle. \tag{4.10}$$

If (4.8) and (4.10) are added together, the left-hand side of the resulting equation is equal and opposite to the rate of working by mixed-layer motions, while the right-hand sides combine to form $\langle\zeta_t p\rangle$. If the terms on the right-hand side of (4.10) are small compared with $\langle\zeta_t p\rangle$, then we may make the approximation

$$\partial E/\partial t \approx \langle\zeta_t p\rangle = -\mathscr{F}, \tag{4.11}$$

which is equivalent to saying that the bulk of the internal wave energy is a result of the work done by the inertial oscillations, rather than the turbulence itself. In general, $u'/U \ll 1$, i.e. the turbulence intensity is small compared with the magnitude of the inertial currents, and we may neglect the triple correlations in (4.10) compared with the terms in (4.8). In effect, a large velocity does proportionately more work against a given force than a small one. In so far as the $\langle w'p\rangle$ correlation is concerned, the arguments concerning eddy persistence in § 3 above indicate that the dominant contribution to ζ_t is $\mathbf{U}\,.\,\nabla\zeta$, so that w' is characteristically small compared with ζ_t. Thus, if u'/U is small, the approximation (4.11) should be valid. It is readily verified that this conclusion remains valid even if the eddy persistence time is of order l/u'.

Equation (4.11) describes the radiation damping of near-surface inertial oscillations. To order u'/U, the rate of decay of the energy of inertial oscillation is equal to the internal wave energy flux. In §§ 2 and 3 above, we have derived expressions for the energy flux in terms of sensible parameters which characterize the upper ocean layer and the spatial structure of the mixed-layer turbulence. The theory is strictly valid only when the amplitude of the inertial oscillation does not vary with time. However, if the characteristic time scale for the wave generation process is small compared with the decay time of the inertial oscillations, then the theory should prove adequate for estimating the decay time. In fact, there are two characteristic times involved in the generation process. The form of the transfer function $\hat{A}$ in (3.24) indicates that the wave-making response is maximized for frequencies of the order of, but somewhat less than, the Brunt–Väisälä frequency (see figure 2 and discussion). Hence we have one characteristic time scale for the generation process which is of order N^{-1}. A necessary, but not sufficient, condition for the validity of the theory is then that the decay time be large compared with N^{-1}. As will be seen, this is normally the case for oceanic applications. However, in deriving manageable expressions for the energy flux, we were forced to average over the phase of the inertial oscillation. If the decay time is large compared with the inertial period, then this averaging procedure is consistent with a slowly decaying inertial oscillation. If, on the other hand, the decay time is not large compared with the inertial period, then we may expect that the decay rate will depend on the initial phase of the inertial oscillation relative to any preferred orientation of the turbulent eddies. In this case, decay rates predicted using the theory presented here must be interpreted as averages for an ensemble of inertial oscillations, rather than as applying to any individual inertial oscillation.

The evolution equation for the ensemble-average amplitude of the inertial oscillations follows directly from (4.11) with $\mathscr{F}$ given by (3.24):

$$\frac{dU_0}{dt} = -\frac{N^2}{\pi D}\int_0^\infty \hat{A}\left(\frac{\kappa U_0}{N}\right)\hat{R}_0(\kappa)\,d\kappa, \tag{4.12}$$

where

$$\hat{A}\left(\frac{\kappa U_0}{N}\right) = \frac{2}{\pi}\frac{\kappa U_0}{N}\int_0^{\phi_0}\sin^2\phi\left(1-\frac{\kappa^2 U_0^2}{N^2}\sin^2\phi\right)^{\frac{1}{2}}d\phi \tag{4.13}$$

with $\phi_0 = \frac{1}{2}\pi$ for $\kappa U_0 < N$ and $\sin\phi_0 = N/\kappa U_0$ otherwise. $\hat{R}_0(\kappa)$ is the spectral function for turbulent displacements of the base of the mixed layer. Equation (4.12) cannot be solved without specifying the spectral function $\hat{R}_0(x)$, which cannot be done at this time owing to a lack of relevant data. However, approximate results can be obtained in certain parameter regimes of interest by invoking appropriate asymptotic approximations. Even an asymptotic evaluation depends to some extent on gross features of the shape of the spectrum, or equivalently the correlation function. We assume that the mixed-layer turbulence is broad-band, i.e. is characterized by a more or less monotonic correlation function, with appreciable correlations restricted to $x < l$, where l is the integral scale defined by

$$\langle\zeta_0^2\rangle l = \int_0^\infty R_0(x)\,dx, \tag{4.14}$$

where $\langle\zeta_0^2\rangle$ is the mean-square displacement of the base of the mixed layer. Although data on the structure of mixed-layer turbulence are limited, it is possible to make order-of-magnitude estimates of the parameters ζ_0 and l. Data from rapid CTD profiling through the mixed layer presented by Pollard (1974) show displacements $\zeta_0 \approx 10^{-1}D$ for mixed-layer depths D of 20–30 m. Precise evaluation of the integral scale l is difficult at best, since spectra of oceanographic variables are characteristically red, with contributions from scales ranging from hundreds or thousands of kilometres down to centimetres. However, since the mixed-layer turbulence is confined within a depth D, we may expect some limitation on the relevant turbulence scales. Indeed, if the mixed-layer turbulence extracts any significant amount of its energy from the mean flow, then we may expect that scales smaller than the mixed-layer depth will dominate the spectrum. In this regard, we note that Dillon & Powell (1976) report mixed-layer turbulence spectra from Lake Tahoe which consistently show a levelling-off of the spectrum for wavelengths greater than about one or two times the mixed-layer depth, i.e. for κD less than about π or 2π. We might also note that observations of Langmuir cells reviewed by Faller (1971) and Assaf, Gerard & Gordon (1971) indicate a characteristic cell spacing roughly equal to the mixed-layer depth. McLeish (1968) suggests that this corresponds to the wavelength characterizing the large eddies of mixed-layer turbulence. All of this evidence tends to suggest that the relevant integral scale is somewhat less than the mixed-layer depth. Indeed, Nihoul (1972) argues on energetics grounds that l should be of the order of $10^{-1}D$, and Niiler (1977), in his review of numerical modelling of the mixed layer, implies that the range

$$0{\cdot}03D < l < 0{\cdot}3D$$

may be appropriate. We shall choose $l \approx D/2\pi$ as representative.

By invoking Parseval's relation, (4.12) may be transformed to

$$\frac{dU_0}{dt} = -\frac{2N^3}{U_0 D}\int_0^\infty A\left(\frac{Nx}{U_0}\right) R_0(x)\,dx, \tag{4.15}$$

where $R_0(x)$ is the correlation function and

$$A\left(\frac{Nx}{U_0}\right) = \frac{1}{\pi}\int_0^\infty \hat{A}(\beta)\cos\left(\frac{Nx}{U_0}\beta\right)d\beta. \tag{4.16}$$

For representative conditions ($N \sim 10^{-2}\,\mathrm{s}^{-1}$, $D \sim 25\,\mathrm{m}$, $U_0 \sim 30\,\mathrm{cm/s}$), we have $Nl/U_0 \approx 0{\cdot}13$. Since appreciable correlations are restricted to $x < l$, we may replace the transfer function $A(Nx/U_0)$ by its limiting form for small argument. $A(Nx/U_0)$ and $\hat{A}(\kappa U_0/N)$ are a Fourier transform pair, so that the behaviour of $A(Nx/U_0)$ for small Nx/U_0 is related to the asymptotic behaviour of $\hat{A}(xU_0/N)$ for large $\kappa U_0/N$ according to

$$\hat{A}(\beta) \sim -(2/\beta^2)A'(0), \tag{4.17}$$

a result which is established by successive partial integrations (see Copson 1967, § 10). From (3.22) we also have, for $\beta \to \infty$,

$$\hat{A}(\beta) \sim \frac{2}{\pi}\beta\int_0^{\beta^{-1}} \phi^2(1-\phi^2\beta^2)^{\frac{1}{2}}\,d\phi = \frac{1}{8\beta^2}. \tag{4.18}$$

Noting that

$$A(0) = \frac{1}{\pi}\int_0^\infty \hat{A}(\beta)\,d\beta = \frac{1}{3\pi}, \tag{4.19}$$

we may combine the results and evaluate terms in the Taylor-series expansion of $A(Nx/U_0)$:

$$\begin{aligned} A(Nx/U_0) &= A(0) + \frac{Nx}{U_0}A'(0) + \frac{1}{2}\left(\frac{Nx}{U_0}\right)^2 A''(0) + \ldots \\ &= \frac{1}{3\pi} - \frac{1}{16}\frac{Nx}{U_0} + O\left(\frac{N^2x^2}{U_0^2}\right). \end{aligned} \tag{4.20}$$

Again, appreciable correlations are restricted to $x < l$, so that

$$\int_0^\infty xR_0(x)\,dx \gtrsim l\int_0^\infty R_0(x)\,dx. \tag{4.21}$$

Substituting the expansion (4.20) into (2.15), we thus obtain the result

$$\frac{dU_0}{dt} = -\frac{2}{3\pi}\frac{N^3 l\langle\zeta_0^2\rangle}{U_0 D}\left\{1+O\left(\frac{Nl}{U_0}\right)\right\} \tag{4.22}$$

for the evolution of the ensemble-average amplitude of the inertial oscillations.

Results valid for all Nl/U_0 may be obtained if we approximate the spectrum $\hat{R}_0(\kappa)$ by its equivalent rectangular spectrum

$$\hat{R}_0(\kappa) = \begin{Bmatrix} 2l\langle\zeta_0^2\rangle & \text{for} & |\kappa| < \pi/2l, \\ 0 & \text{for} & |\kappa| > \pi/2l. \end{Bmatrix} \tag{4.23}$$

In this case, the evolution equation (4.12) becomes

$$\frac{dU_0}{dt} = -\frac{2}{3\pi}\frac{N^3 l\langle\zeta_0^2\rangle}{U_0 D}\left\{1-\frac{2}{\pi}\int_0^{\phi_0}\left(1-\frac{\pi^2 U_0^2}{4N^2l^2}\sin^2\phi\right)^{\frac{3}{2}} d\phi\right\}, \tag{4.24}$$

where $\phi_0 = \frac{1}{2}\pi$ for $Nl/U_0 > \frac{1}{2}\pi$ and $\sin\phi_0 = 2Nl/\pi U_0$ for $Nl/U_0 < \frac{1}{2}\pi$. By introducing a scaled, non-dimensional time

$$\tilde{t} = \langle\zeta_0^2\rangle Nt/lD \tag{4.25}$$

and velocity

$$\tilde{U} = \tfrac{1}{2}\pi U_0/Nl \tag{4.26}$$

(2.24) may be written in the form

$$-\frac{1}{\tilde{U}}\frac{d\tilde{U}}{d\tilde{t}} = \frac{\pi}{6\tilde{U}^2}\left\{1-\frac{2}{\pi}\int_0^{\phi_0}(1-\tilde{U}^2\sin^2\phi)^{\frac{3}{2}}\,d\phi\right\}. \tag{4.27}$$

The quantity

$$S(\tilde{U}) = -\tilde{U}^{-1}d\tilde{U}/d\tilde{t} \tag{4.28}$$

is a non-dimensional decay rate for the ensemble-average amplitude of the inertial oscillations. In a time interval $\delta\tilde{t}$, $\tilde{U}$ will decrease by a fraction $S\,\delta\tilde{t}$. For $\tilde{U} \gg 1$, $\sin\phi \approx \phi$ and we have

$$\begin{aligned}-\frac{1}{\tilde{U}}\frac{d\tilde{U}}{d\tilde{t}} &\sim \frac{\pi}{6\tilde{U}^2}\left\{1-\frac{2}{\pi}\int_0^{\tilde{U}^{-1}}(1-\tilde{U}^2\phi^2)^{\frac{3}{2}}\,d\phi\right\}\\ &= \frac{\pi}{6\tilde{U}^2}\left\{1-\frac{3}{8\tilde{U}}\right\}.\end{aligned} \tag{4.29}$$

It is readily verified that this result may also be obtained directly from (4.15) with $A(Nx/U_0)$ given by (4.20), provided that the integral of $xR_0(x)$ is treated in the sense of generalized functions (see Lighthill 1964, § 3.2). For arbitrary $\tilde{U}$, the integral in (4.27) may be evaluated in terms of tabulated elliptic integrals. It should be noted, however, that with $N \approx 10^{-2}\,\text{s}^{-1}$ and $D \approx 25\,\text{m}$, $\tilde{U} = 1$ corresponds to a velocity $U_0 \approx 2{\cdot}5\,\text{cm/s}$, which is likely to be comparable to the turbulence intensity u'. Since the theory developed above is strictly valid only for $U_0/u' \gg 1$, the behaviour of the decay rate for $\tilde{U}$ the order of, or less than, unity is irrelevant for oceanographic applications.

The non-dimensional decay rate $S(\tilde{U})$ for the equivalent rectangular spectrum is illustrated as a function of $\tilde{U}$ in figure 3. Also shown is the asymptotic approximation

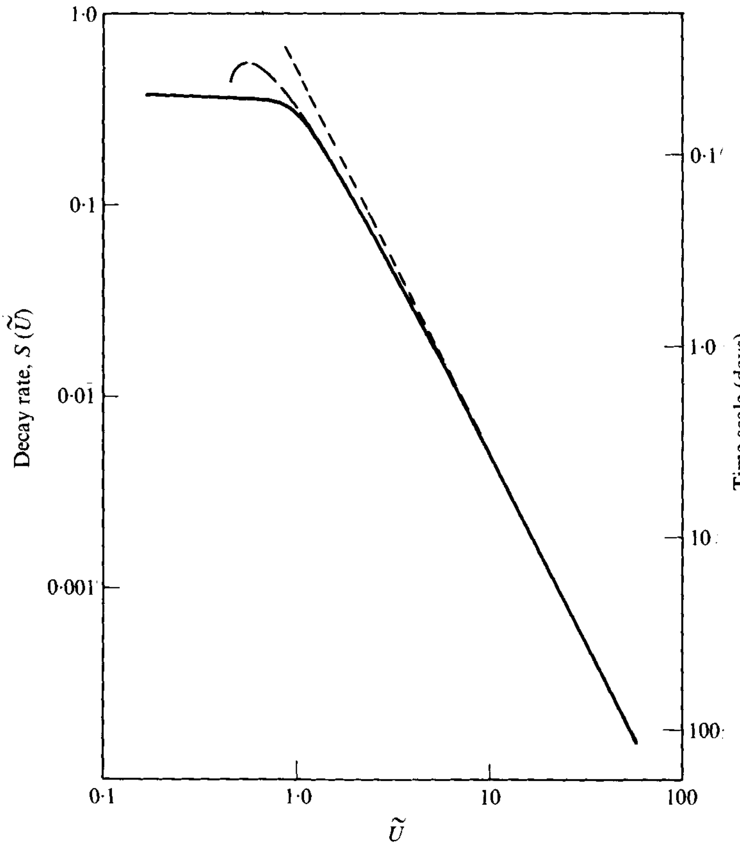

FIGURE 3. The non-dimensional decay rate S for radiation damping of inertial oscillations as a function of the non-dimensional amplitude $\tilde{U}$ of the motions [see (4.36)–(4.39)]. ——, exact result for equivalent rectangular spectrum; ——, approximate result given by (4.40); ---, leading term for large $\tilde{U}$. Corresponding dimensional time scale is indicated on right-hand side, under the assumption that $N \approx 10^{-2}\,\mathrm{s}^{-1}$.

(4.29) and its leading term. For $\tilde{U} > 1$, the leading term in (4.29) is within a factor of two of the exact result, and is virtually indistinguishable from it by the time $\tilde{U}$ is near 10. The corresponding dimensional time scale is indicated on the right of figure 3 under the assumption that $\zeta_0 \approx 10^{-1}D$, $l \approx D/2\pi$ and $N = 10^{-2}\,\mathrm{s}^{-1}$. For velocities U_0 ranging from 10 to 50 cm/s, with $N = 10^{-2}\,\mathrm{s}^{-1}$ and $D = 25\,\mathrm{m}$, the relevant time scales range from just under a day to about two weeks, which is consistent with the observed range of time scales characterizing the decay of near-surface inertial oscillations (see, for example, Webster 1968; Pollard & Millard 1970; Gonella 1971; Halpern 1974; Hayes & Halpern 1976).

5. The internal wave field

From (4.11) and (4.22), the flux of energy into the internal wave field is given by

$$\mathscr{F} = \frac{2}{3\pi}\rho_0 N^3 l \langle \zeta_0^2 \rangle \left\{1 + O\left(\frac{Nl}{U_0}\right)\right\} \tag{5.1}$$

for $U_0 \gg Nl$, where N is the Brunt–Väisälä frequency, l is the integral scale of the mixed-layer turbulence, ζ_0 is the r.m.s. turbulent displacement of the base of the mixed layer, and U_0 is the magnitude of the inertial oscillation. For reasons noted in the preceding section, we may expect that in general the leading term in (5.1) will provide a viable approximation to the energy flux provided that $U_0 \gtrsim 5$ cm/s. If we assume, as before, that $\zeta_0 \approx 10^{-1}D$ and $l \approx D/2\pi$, where D is the mixed-layer depth, then for representative conditions ($N \approx 10^{-2}\,\mathrm{s}^{-1}$, $D \approx 25$ m) (5.1) gives $\mathscr{F} \approx 5\,\mathrm{erg/cm^2 s}$. The magnitude of this energy flux suggests that this mechanism may represent a significant source of internal wave energy in the upper ocean, since, according to Müller & Olbers (1975), any internal wave energy-transfer rates of order $1\,\mathrm{erg/cm^2 s}$ or larger are potentially significant in the overall internal wave energy budget. Since the inertial oscillations are intermittent, internal wave energy is not necessarily being continuously generated at this rate. In figure 4, we have plotted a sample probability distribution function of U_0 on logarithmic probability paper, using data on the inertial period amplitude of near-surface currents from Webster (1968), Pollard & Millard (1970) and Halpern (1974). If we assume that these data are representative, then U_0 is above a threshold of 5 cm/s about 75 % of the time, so that a long-term average energy flux of about $4\,\mathrm{erg/cm^2 s}$ might be considered as representative. This is still potentially significant in so far as the overall internal wave energy budget is concerned. The theory developed here is incapable of adequately describing the internal wave generation process for $U_0 \lesssim 5$ cm/s, although we may infer that internal waves are still being generated at the lower speeds. Indeed, it is possible that continuous low-level forcing by the residual currents (including the turbulent motions thermselves) may also be significant.

Some idea of the distribution of internal wave energy in wavenumber–frequency space may be obtained from consideration of equation (2.18) for the energy density. With the turbulence correlation function specified by (3.10), (3.11) and (3.23), the spectral function for the internal wave energy density is given by

$$\hat{\mathscr{E}}(\kappa, \omega) = 2\pi\rho_0\omega^2 \frac{N^2-f^2}{\omega^2-f^2} \Sigma J_n^2\left(\frac{\kappa U_0}{f}\right)\hat{R}_0(\kappa)\,\delta(\omega-nf). \tag{5.2}$$

The singularity at $\omega = f$ is somewhat artificial. Since the vertical group velocity [see (2.9)] vanishes at the inertial frequency, energy at $\omega = f$ would never be seen at an observation point below the mixed layer. Indeed, since we are dealing with an intermittent phenomenon, the group velocity is an important factor to be borne in mind when considering the energy density. From (2.10) and (2.11), the magnitude of the vertical group velocity is given by

$$|\gamma| = \frac{1}{|\omega|\,\kappa}\frac{\omega^2-f^2}{N^2-f^2}[(N^2-\omega^2)(\omega^2-f^2)]^{\frac{1}{2}}. \tag{5.3}$$

$|\gamma|$ approaches zero as ω approaches f or N, and is maximal for

$$\omega_0^2 = \tfrac{1}{3}N^2\{1+(1+3f^2/N^2)^{\frac{1}{2}}\}. \tag{5.4}$$

For $N^2 \gg f^2$, $\omega_0 = \pm(\frac{2}{3})^{\frac{1}{2}}N$. At any finite time after the onset of internal wave generation, the observed wave field will cover a band of frequencies more or less centred about ω_0. At any given depth below the mixed layer, the first waves to be observed will

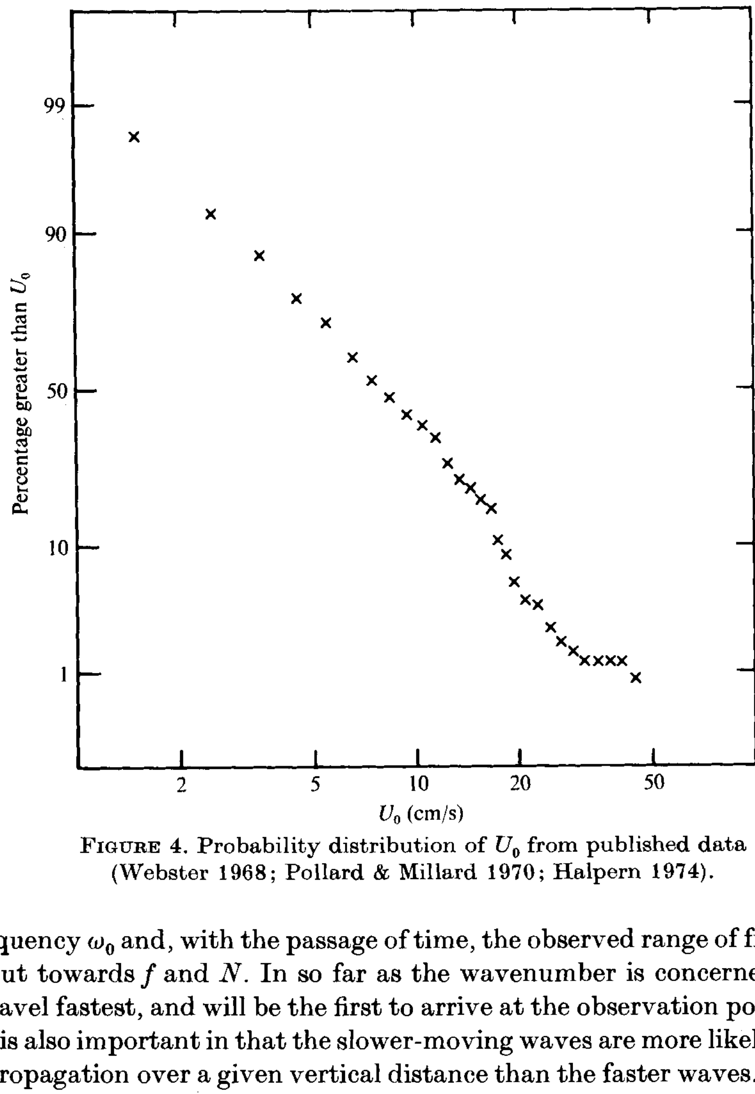

FIGURE 4. Probability distribution of U_0 from published data (Webster 1968; Pollard & Millard 1970; Halpern 1974).

be of frequency ω_0 and, with the passage of time, the observed range of frequencies will spread out towards f and N. In so far as the wavenumber is concerned, the longest waves travel fastest, and will be the first to arrive at the observation point. The group velocity is also important in that the slower-moving waves are more likely to be altered during propagation over a given vertical distance than the faster waves. The energy in the slower waves may be transferred to other spectral ranges by nonlinear interactions. Also, the slower waves have proportionately smaller vertical scales and are more susceptible to scattering by fine structure in the background density and velocity fields than are the faster waves.

Referring back to (5.2), if we 'smear' the δ-function, the energy spectrum is given by

$$\hat{\mathscr{E}}(\kappa, \omega) \approx \rho_0 N^2 J^2_{\omega/f}(\kappa U_0/f)\,\hat{R}_0(\kappa) \tag{5.5}$$

for $\omega \gg f$, consistent with the arguments concerning the group velocity. The smeared δ-function is probably more realistic than (5.2) in that it allows for some random time dependence in the turbulence, and is appropriate to high frequency waves. Recalling (3.13), it is clear that the dominant contribution to the spectrum as defined by (5.5) will occur for $\omega \approx \kappa U_0$ when $\omega \gg f$, provided that $\hat{R}_0(\kappa)$ is reasonably well behaved. In the present context, $\hat{R}_0(\kappa)$ may be considered well behaved if $\kappa \gtrsim l^{-1}$, where l is the integral scale of the turbulence. Thus we expect that the radiated wave field will be characterized by relatively high frequencies (of the order of, but somewhat less than, the Brunt–Väisälä frequency) and short wavelengths (of the order of $2\pi U_0/N$, which may be as large as several hundred metres). The amplitude of the waves will be comparable to the amplitude of the vertical displacement of the base of the mixed layer, which, as noted previously, may be several metres.

6. Discussion

We have considered the generation of internal waves by the interaction of mixed-layer turbulence advected by near-surface inertial oscillations with the underlying stratified upper ocean layers. Using an idealized model, which, we hope, retains the essential physics of the interaction process, we have been able to obtain rather simple expressions for the rate of generation of internal wave energy and the characteristic time scale for the decay of energy in near-surface inertial oscillations. For representative conditions ($N \approx 10^{-2}\,s^{-1}$, mixed-layer depth $D \approx 25$ m, current magnitude $U_0 \approx 25$ cm/s), we estimate an energy flux of 3 erg/cm^{2}s into the internal wave field, resulting in a characteristic time scale of about 3·4 days for the decay of the inertial current speed (twice the energy decay time scale). Since the precise values of the parameters are somewhat uncertain, it is probably more appropriate to extend these estimates to probable ranges, to wit, 1–10 erg/cm^{2}s for the energy flux and a fraction of a day to several days for the energy decay time scale.

Although these are only order-of-magnitude estimates, they do suggest two important conclusions. First, it would appear that this mechanism is capable of accounting for the observed transience of near-surface inertial oscillations and, second, this may represent a significant source of internal wave energy in the upper ocean, since according to Müller & Olbers (1975) any internal wave energy-transfer rates of order 1 erg/cm^{2}s or larger are potentially significant in the overall internal wave energy budget. The identification of this mechanism as a viable agent for inertial oscillation transience rectifies a longstanding gap in our understanding of the dynamics of inertial oscillations (see Smith 1973). In so far as internal wave energetics are concerned, this mechanism is of particular interest in that the generated waves will characteristically be of rather high frequency (a significant fraction of the Brunt–Väisälä frequency), and will tend to be intermittent, just as the inertial oscillations from which they derive their energy are. Intermittent groups of high frequency internal waves are characteristic of the upper ocean in general.

Owing to a lack of detailed information on the structure of mixed-layer turbulence, we have been forced to adopt an idealized model and resort to asymptotic evaluation techniques. It should be possible, however, to conduct definitive experiments to test the validity of the theory. A conclusive experiment should involve three components. First, a towed thermistor chain extending through the mixed layer should provide the required estimates of the r.m.s. displacement ζ_0 and the integral scale l. Second, CTD profiles would permit the specification of the appropriate Brunt–Väisälä frequency N, as well as providing supplementary information on ζ_0 and, of course, specifying the mixed-layer depth D. Finally, moored current meters within and below the mixed layer would provide the necessary information on the relative motion between the mixed layer and the underlying ocean layers. A measurement programme incorporating these components would go far in verifying the theory presented here, and it is believed that the potential significance of the results reported here justify the undertaking of such an experiment.

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